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Front formation and motion in quasilinear parabolic equations

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Abstract

This paper deals with the singular limit for

$$\mathcal{L}^\varepsilon u := u_t - F(u, \varepsilon u_x)_x - \varepsilon^{-1} g(u) = 0,$$

where the function F is assumed to be smooth and uniformly elliptic, and g is a "bistable" nonlinearity. Denoting with u_m the unstable zero of g, for any initial datum u_0 for which $u_0 - u_m$ has a finite number of zeroes, and $u_0 - u_m$ changes sign crossing each of them, we show the existence of solutions and describe the structure of the limiting function $u^0 = \lim_{\varepsilon\to 0^+} u^\varepsilon$, where u^ε is the solution of a corresponding Cauchy problem. The analysis is based on the construction of travelling waves connecting the stable zeros of g and on the use of a comparison principle.

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1. Introduction

In this paper we deal with the study of the singular limit for the scalar one-dimensional equation

$$\mathcal{L}^{\varepsilon} u := u_t - F(u, \varepsilon u_x)_x - \varepsilon^{-1} g(u) = 0, \tag{1}$$

where $u = u(x,t) \in \mathbb{R}$, $(x,t) \in \mathbb{R} \times (0,\infty)$ and $\varepsilon > 0$ is a positive parameter. We assume on $F = F(u,p)$ the *uniform ellipticity condition*, that is

(H1) $F \in C^2$, $F_p(u,p) \geqslant \nu > 0$ for all u, p.

An interesting case is $F(u,p) = \nu p - f(u)$, in which we get the so-called reaction–diffusion–convection equation

$$\mathcal{L}^{\varepsilon} u := u_t - \varepsilon \nu u_{xx} + f(u)_x - \varepsilon^{-1} g(u) = 0. \tag{2}$$

Let u^{ε} be the unique solution of the Cauchy problem for (1), given by the initial condition

$$u(x,0) = u_0(x), \tag{3}$$

then the problem is to determine the existence and the structure of the limiting function

$$u^0 = \lim_{\varepsilon \to 0^+} u^{\varepsilon} \quad \text{a.e. in } \Sigma_T = \mathbb{R} \times [0,T],$$

for any $T > 0$. Throughout the paper, we assume that the reaction function g belongs to $C^1(\mathbb{R})$ and satisfies

(H2) g possesses exactly three zeroes $u_\ell < u_m < u_r$ with $g'(u_\ell) < 0$, $g'(u_m) > 0$, $g'(u_r) < 0$.

Theorem 1. *Assume* (H1)–(H2). *Let u^{ε} be the solution of the Cauchy problem*

$$u_t = F(u, \varepsilon u_x)_x + \varepsilon^{-1} g(u), \quad u(x,0) = u_0(x) \tag{4}$$

for some initial data $u_0 \in L^{\infty}(\mathbb{R},\mathbb{R}) \cap H^s_{\text{loc}}(\mathbb{R},\mathbb{R})$ for some $s > 2$ satisfying the following assumption:

(H3) *there exists $\{x_1 < x_2 < \cdots < x_N\} \subset \mathbb{R}$ for which*:
 (i) *for any $\delta > 0$, sufficiently small, and for any $k = 0, \dots, N$,*

$$\begin{aligned} &\textit{either} \quad \operatorname{ess\,sup}\{u_0(x)\colon x \in (x_k + \delta, x_{k+1} - \delta)\} < u_m, \\ &\textit{or} \quad\quad \operatorname{ess\,inf}\{u_0(x)\colon x \in (x_k + \delta, x_{k+1} - \delta)\} > u_m, \end{aligned}$$

 where $x_0 = -\infty$ and $x_{N+1} = +\infty$;
 (ii) *for any $k = 1, \dots, N$, if $u_0(x) < u_m$ a.e. in (x_{k-1}, x_k), then $u_0(x) > u_m$ a.e. in (x_k, x_{k+1}).*

Then $u^0(x,t) := \lim_{\varepsilon \to 0} u^{\varepsilon}(x,t)$ exists for a.e. (x,t). More precisely, $u^0(x,t)$ is a piecewise constant function taking only the values u_ℓ and u_r. There exist $c_-, c_+ \in \mathbb{R}$ such that *regions where u^0 is constant are separated by straight lines $x = x_k + ct$, where $c = c_-$ if $u_0(x) < u_m$ a.e. in (x_k, x_{k+1}), $c = c_+$ if $u_0(x) > u_m$ a.e. in (x_k, x_{k+1}), and $0 \leqslant t \leqslant T_k$. Here T_i is either $+\infty$ or finite.*

Assumption (H2) implies that the states u_ℓ and u_r are stable with respect to the o.d.e. $u' = g(u)$, while u_m is unstable. Assuming only g smooth, with simple zeros and an assumption similar to (H3), an analogous result can be proved: the singular limit u^0 is a piecewise constant function taking only the values in the set of stable zeros, separated by straight lines. The slope of the straight lines is given by the speed of appropriate traveling waves of the reaction–diffusion–convection equation. Here we restrict ourselves to the case of three zeros for clarity of presentation. This assumption permits to know a priori that straight lines separate regions in which $u^0 = u_\ell$ from regions in which $u^0 = u_r$.

The singular perturbation problem (1) comes into view when performing a hyperbolic rescaling $(x, t) \to (x/\varepsilon, t/\varepsilon)$ in the equation

$$u_t = F(u, u_x)_x + g(u).$$

The singular limit $\varepsilon \to 0$ gives a representation of the large-time behavior of the solution. Since also the space variable x is rescaled the limiting function u^0 encodes the behavior of the solution of the original problem observed from far distance also. For this reason, this kind of approach is not able to catch the inner structure of the transition from one stable state to the other, and only gives sharp jumps and sharp cancellation, when two different transitions interact. The most interesting fact is that the structure of u^0 is determined, apart from the initial datum u_0, only by the speeds of propagation of the travelling waves that describes the connections between the stable states u_ℓ, u_r. This situation is already present in the simpler case of reaction–diffusion equation

$$u_t = \varepsilon u_{xx} + \varepsilon^{-1} g(u), \tag{5}$$

for which a wide literature is available (see [1] and references therein). The main difference in the case we consider, is that problem (1) is no longer isotropic, while (5) is. This means that two kind of transitions from one stable state to the other are present, depending on which one is attained at the left and which one at the right. Anisotropy is much more evident thinking of the multidimensional case: while for (5) radially symmetric initial data give raise to radially symmetric solutions, for (1) this is not true anymore. Nevertheless, there is still a very strong analogy between the reaction–diffusion case and (1), that is the existence and the stability of travelling waves, which encodes all of the properties of the transitions.

It has to be stressed that the same kind of phenomenon is present for the hyperbolic reaction–convection equation:

$$u_t + f(u)_x = \varepsilon^{-1} g(u). \tag{6}$$

Even if the regularity of the solutions to (6) is very different (no smoothing effects, shock formation, entropy solution...), a picture similar to the one we show in the present paper has been found in [4,8]. Moreover, the result in [8] is based on the construction of (entropy) travelling waves describing transitions from stable states and by using a comparison principle, in the same spirit as we do here, but in a different framework. Since the class (1) contains also reaction–diffusion–convection equations of the form (2), an interesting link between (5) and (6) is established by Theorem 1. Recently, in [3] a multi-dimensional version of (2) was studied for the case of a symmetric flux function f and an odd source term g. In this situation the front motion is dominated by the mean curvature. In general, however, the velocity of a front in (1) would depend on the normal direction and therefore lead to anisotropic front motion.

To prove Theorem 1 we have to consider two steps of the dynamics. First, due to the stiff source term the solution will approach a step function where almost constant pieces are separated by steep layers. Then these layers move and may collide. The movement of the layers is determined mainly by travelling wave solutions connecting the corresponding asymptotic states. For this reason, we provide an existence and uniqueness result for travelling waves of the quasilinear equation (1). The existence Theorem 5 states that, fixing some stable asymptotic states of the wave, there is exactly one wave speed and one profile connecting them. This is due to the fact that the travelling wave corresponds to a saddle–saddle connection, as in the reaction–diffusion case. These travelling waves are then used to construct sub- and supersolutions with one or several moving layers and applying comparison principle for (1) as stated in [11].

Let us briefly comment on the general existence and uniqueness for solutions of (1). Classical assumptions that are sufficient to guarantee unique solutions for quasilinear equations on $\mathbb{R}^n$ are, for instance, given in the classical book [7]. There it is shown that under a uniform ellipticity condition and assuming some regularity on the coefficients (which is satisfied in our case if $F \in C^2$) for any sufficiently regular, bounded initial condition there will be a unique bounded solution of the Cauchy problem which lies locally in some Sobolev space H^s with $s > 2$.

The paper is organized in the following way: in Section 2 we establish the existence of travelling waves, Section 3 deals with the formation of steep layers near points where u_0^ε changes its sign and in Section 4 we construct sub- and supersolution with moving layers to prove the main result.

2. Existence of traveling waves

Looking for solution of the form $u(x,t) = U(x - ct/\varepsilon)$, we get the ordinary differential equation

$$F_p(U, U')U'' + \big(c + F_u(U, U')\big)U' + g(U) = 0,$$

or, in the phase plane,

$$\begin{aligned} U' &= V \\ V' &= h(U, V) := -\frac{(c + F_u(U, V))V + g(U)}{F_p(U, V)}. \end{aligned} \tag{7}$$

Note that this system is a *rotated vector field* (mod $V = 0$) with respect to the parameter c, in the sense introduced by Duff [2], see also [9,10]:

Definition 2. A planar vector field

$$U' = P(U, V, c), \qquad V' = Q(U, V, c)$$

depending on some scalar parameter c is called a *rotated vector field* $(\operatorname{mod} G = 0)$ if the equilibrium points remain the same for all values of the parameter c and the determinant

$$\begin{vmatrix} P(U, V, c) & Q(U, V, c) \\ \frac{\partial P(U,V,c)}{\partial c} & \frac{\partial Q(U,V,c)}{\partial c} \end{vmatrix} < 0$$

for all U, V for which $G(U, V) \neq 0$, where G is some analytic function.

Lemma 3. *The travelling wave vector field* (7) *is a rotated vector field* $(\operatorname{mod} V = 0)$ *with respect to the wave speed* c.

Proof. This follows by direct computation:

$$\begin{vmatrix} V & -\frac{(c+F_u(U,V))V+g(U)}{F_p(U,V)} \\ 0 & -\frac{V}{F_p(U,V)} \end{vmatrix} = -\frac{V^2}{F_p(U, V)} < 0$$

whenever $V \neq 0$. □

Given any trajectory of (7) at a fixed parameter value c_0, this implies that for $c > c_0$ the vector field will cross this trajectory from one side to the other while for $c < c_0$ the crossing will be in the opposite direction.

Rotated vector fields possess the property that the invariant manifolds of all saddle equilibria "rotate" in the same direction as the parameter is varied. More precisely:

Proposition 4 [10, Theorem 5]. *Let S be a separatrix of a saddle equilibrium of a rotated vector field. Assume that S intersects a curve L which for all value of the parameter c is transverse to the vector field. Then the intersection point varies monotonically with c along the curve L.*

This can be used to prove the existence of a unique parameter value (in our case, unique wave speed) for which a heteroclinic connection between two saddle equilibria exists. The singular points are given by

$$U \in \{u_\ell, u_m, u_r\}, \qquad V = 0,$$

and the linearized system at $(u^*, 0)$ is

$$U' = V, \qquad V' = -\frac{g'(u^*)}{F_p(u^*, 0)}(U - u^*) - \frac{c + F_u(u^*, 0)}{F_p(u^*, 0)} V.$$

The eigenvalue equation is

$$F_p(u^*, 0)\mu^2 + \big(c + F_u(u^*, 0)\big)\mu + g'(u^*) = 0,$$

and the eigenvalues

$$\mu_\pm(c) = \frac{-(c + F_u(u^*,0)) \pm \sqrt{(c + F_u(u^*,0))^2 - 4F_p(u^*,0)g'(u^*)}}{2F_p(u^*,0)}.$$

Hence the two stable zeros u_ℓ and u_r are always saddles. The corresponding eigenvectors are

$$e_\pm = \begin{pmatrix} 1 \\ \mu_\pm \end{pmatrix}.$$

Asymptotically, we have

$$\mu_+(c) = -\frac{g'(u^*)}{c + F_u(u^*,0)} + \mathcal{O}(c^{-2}) \quad \text{for } c \to +\infty,$$
$$\mu_-(c) = -\frac{c + F_u(u^*,0)}{F_p(u^*,0)} + \mathcal{O}(1) \quad \text{for } c \to +\infty,$$
$$\mu_+(c) = -\frac{c + F_u(u^*,0)}{F_p(u^*,0)} + \mathcal{O}(1) \quad \text{for } c \to -\infty,$$
$$\mu_-(c) = -\frac{g'(u^*)}{c + F_u(u^*,0)} + \mathcal{O}(c^{-2}) \quad \text{for } c \to -\infty.$$

We denote with $W_\ell^u(c)$ the unstable manifold of $(u_\ell, 0)$ and with $W_r^s(c)$ the stable manifold of $(u_r, 0)$.

Theorem 5. *There exists a unique wave speed c_+ such that for $c = c_+$ there is a heteroclinic orbit of* (7) *connecting $(u_\ell, 0)$ to $(u_r, 0)$. The corresponding travelling wave profile U_+ is monotone increasing. Similarly, there is a unique wave speed c_- for which a travelling wave with monotone decreasing profile U_- connecting $(u_r, 0)$ to $(u_\ell, 0)$ exists.*

Proof. Both statements can be proved in the same way, so we only show that there is a travelling wave from $(u_\ell, 0)$ to $(u_r, 0)$. In this case we have to show that $W_\ell^u(c_+) \cap W_r^s(c_+) \neq \emptyset$ for some wave speed c_+. To this end we choose some constant $k > 0$ and evaluate the vector field along the line $V = k(U - u_\ell)$. The slope of the vector field is

$$\frac{V'}{U'} = F_p^{-1}(U, V)\left(-c - F_u(U, V) - \frac{g(U)}{k(U - u_\ell)}\right).$$

Since both F_p and F_u are bounded in the compact triangle

$$\{(U, V);\ u_\ell \leqslant U \leqslant u_r,\ 0 \leqslant V \leqslant k(U - u_\ell)\}$$

it is possible to achieve that

$$\frac{V'}{U'} > k$$

along the line $V = k(U - u_\ell)$ for $c = \underline{c}$, where $-\underline{c}$ is sufficiently large. In other words, trajectories cross that line from below. Since the tangent vector e_+ of the unstable manifold $W_\ell^u(\underline{c})$ has the slope $\mu_-(u_\ell)$ we see that for $\underline{c}$ sufficiently large W_ℓ^u lies above the line

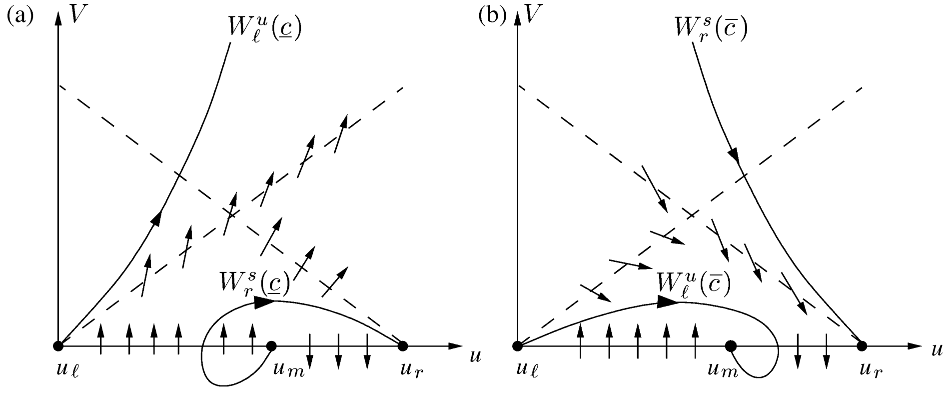

Fig. 1. (a) The phase portrait for $-c$ large, (b) the phase portrait for large c.

$V = k(U - u_\ell)$. A similar calculation shows that the stable manifold W_r^s of $(u_r, 0)$ lies below the line $V = -k(U - u_r)$. The situation is depicted in Fig. 1a. For wave speeds larger than some number $\bar{c}$ the same argumentation shows that W_ℓ^u lies below the line $V = k(U - u_\ell)$ if c is sufficiently large while W_r^s lies above the line $V = -k(U - u_r)$, see Fig. 1b. Increasing $\bar{c}$, if necessary, we may assume that $V' < 0$ along the line $V = k(u_r - u_\ell)$ for $c = \bar{c}$. Similarly, we may assume that $V' > 0$ along the line $V = k(u_r - u_\ell)$ for $c = \underline{c}$.

Typically, one would like to measure somehow the distance between W_ℓ^u and W_r^s for values $c \in [\underline{c}, \bar{c}]$ to show that there is some intersection of the two manifolds. However, there is no obvious choice of some line which is transverse to the vector field for all c and which would allow to define the distance as the distance of the intersections of W_ℓ^u and W_r^s with this line. For instance, without additional assumptions on F the unstable manifold W_ℓ^u may not intersect the line $U = u_m$ for $-c$ sufficiently large.

The most difficult part will therefore consist of the proof that for some c both $W_\ell^u(c)$ and $W_r^s(c)$ do intersect some vertical line $U = \text{const}$.

To this end, we choose δ such that $W_r^s(\bar{c})$ intersects the vertical line $L_\delta := \{(U, V);\ U = u_r - \delta,\ V > 0\}$. Since we have a rotated vector field and any vertical line is transverse to the vector field for $V \neq 0$ and all c, by Proposition 4 the intersection of $W_r^s(c)$ with L_δ varies monotonically with c. We define $\Theta_r(c)$ to be the V coordinate of this intersection. Then $\Theta_r(c)$ is monotone increasing and positive for $c \in [\underline{c}, \bar{c}]$.

To show that $W_\ell^u(c)$ intersects L_δ for some value $c \in [\underline{c}, \bar{c}]$ we proceed in three steps.

For $(U, V) \in Q := [u_\ell, u_r] \times [V_{\min}, V_{\min} + 1]$ the slope of trajectories

$$\left|\frac{V'}{U'}\right| = \frac{\sup_Q |F_u(U, V)| + \max\{|\underline{c}|, |\bar{c}|\}}{\inf_Q |F_p(U, V)|} + \frac{\sup_Q |g(U)|}{\inf_Q |F_p(U, V)| V_{\min}}$$

is bounded. This implies that we can find $\eta > 0$ small such that every trajectory that passes through the point $(u_m - \eta, V_{\min})$ for some $c \in [\underline{c}, \bar{c}]$ will intersect the line $L_+ := \{U = u_m + \eta\}$.

Observe first that $W_\ell^u(\bar{c})$ intersects the line $L_- := \{U = u_m - \eta\}$. Both lines L_- and L_+ are transverse to the vector field for all $c \in [\underline{c}, \bar{c}]$. By Proposition 4 the point of inter-

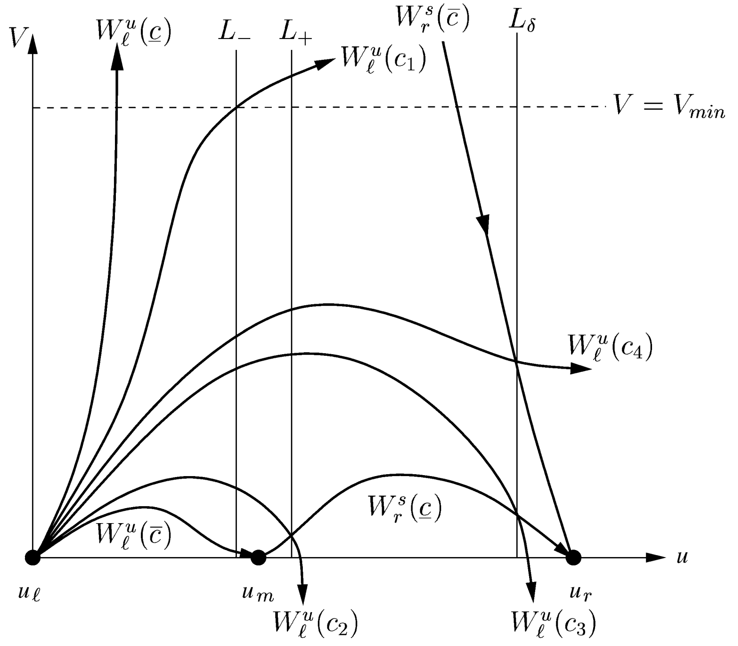

Fig. 2. The invariant manifolds for different values of c.

section between $W_\ell^u(c)$ and L_- moves upwards as c decreases until it reaches the point $(u_m - \eta, V_{\min})$ at $c_1 \in (\underline{c}, \bar{c})$. By our choice of η this implies that $W_\ell^u(c_1)$ also intersects L_+. If we now increase c again the point of intersection between $W_\ell^u(c_1)$ and L_+ moves down until at $c = c_2 \in (c_1, \bar{c})$ we have

$$W_\ell^u(c_2) \cap L_+ = W_r^s(\underline{c}) \cap L_+,$$

see Fig. 2.

Then we decrease c again. Since $W_r^s(\underline{c})$ is transverse to the vector field (7) for all $c \neq \underline{c}$ the point of intersection between $W_\ell^u(c)$ and $W_r^s(\underline{c})$ moves to the right until for $c = c_3 \in (\underline{c}, c_2)$.

In particular, we have found a wave speed c_3 such that both $W_\ell^u(c_3)$ intersects the line L_δ.

In a last step we decrease c such that the point of intersection between $W_\ell^u(c)$ and L_δ moves up and reaches the point $W_r^s(\bar{c}) \cap L_\delta$ at $c = c_4$.

For $c \in [c_3, c_4]$ we let $\Theta_\ell(c)$ be the V-coordinate of the intersection between W_ℓ^u and L_δ. Then Θ_ℓ is a continuous, monotone decreasing function with

$$\Theta_\ell(c_3) = \Theta_r(\underline{c}), \qquad \Theta_\ell(c_4) = \Theta_r(\bar{c}).$$

Since $\underline{c} < c_4 < c_3 < \bar{c}$ it follows now from the intermediate value theorem that there exists some c_+ for which $\Theta_\ell(c_+) = \Theta_r(c_+)$ and hence $W_\ell^u(c_+) \cap W_r^s(c_+) \neq \emptyset$.

Uniqueness of the wave speed c_+ follows from the fact that (7) is a rotated vector field. Both the unstable manifold of $(u_\ell, 0)$ and the stable manifold of $(u_r, 0)$ rotate clockwise as c is decreased. This implies that the connection splits under a small change of the wave speed because $W_\ell^u(c)$ and $W_r^s(c)$ move in opposite directions. The corresponding profile U_+ is monotone increasing as it lies completely in the half plane $\{V > 0\}$. □

Lemma 6. *Theorem* 5 *continues to hold if $g(u)$ is replaced by $g(u) + \lambda$ with $|\lambda|$ sufficiently small, i.e., there is a unique wave speed c_+^λ and a unique monotone increasing profile U_+^λ connecting two saddle equilibria u_ℓ^λ and u_r^λ close to u_ℓ and u_r. Similarly, there is a unique monotone decreasing profile U_-^λ from u_r to u_ℓ. The corresponding wave speeds satisfy*

$$\left|c_\pm^\lambda - c_\pm\right| = \mathcal{O}(\lambda). \tag{8}$$

Proof. For $|\lambda|$ sufficiently small, $g(u) + \lambda$ still possesses three zeroes $u_\ell^\lambda < u_m^\lambda < u_r^\lambda$ with $g'(u_\ell^\lambda) < 0$, $g'(u_m^\lambda) > 0$ and $g'(u_r^\lambda) < 0$.

Thus, from the proof of Theorem 5 we know that there is a unique wave speed c_+^λ and a unique profile U_+^λ.

To prove (8) we employ a standard Melnikov calculation, see [5]. To this end we note that if we measure the distance $\rho(c, \lambda)$ between $W_\ell^u(c, \lambda)$ and $W_r^s(c, \lambda)$ along a vertical section then the derivative of ρ with respect to the parameters c and λ is given by the Melnikov integrals

$$\frac{\partial \rho}{\partial c}\left(c_+^0, 0\right) = \int_{-\infty}^{+\infty} \psi(s)^T \begin{pmatrix} 0 \\ V_+(s)/F_p(U_+(s), U_+'(s)) \end{pmatrix} ds,$$

$$\frac{\partial \rho}{\partial \lambda}(c_+, 0) = \int_{-\infty}^{+\infty} \psi(s)^T \begin{pmatrix} 0 \\ 1/F_p(U_+(s), U_+'(s)) \end{pmatrix} ds.$$

Note that ψ is the (suitably scaled) unique bounded solution of the adjoint variational equation and $U_+(s)$ is the unperturbed heteroclinic for $\lambda = 0$. Since $(U_+', V_+')^T$ solves the linearized equation, ψ is always perpendicular to $(U_+', V_+')^T$. For the monotone increasing profile U_+ we know that U_+' is positive which implies that the second component of ψ is also positive. This in turn implies that both of the Melnikov integrals are negative and by the implicit function theorem

$$\frac{dc_+^\lambda}{d\lambda}(0) = -\frac{\frac{\partial \rho}{\partial c}(c_+, 0)}{\frac{\partial \rho}{\partial \lambda}(c_+, 0)} < 0$$

is finite.

Similarly, for the monotone decreasing profile U_-^λ with wave speed c_-^λ we have $V_- < 0$. The second component of ψ is negative, too, so

$$\frac{dc_-^\lambda}{d\lambda}(0) = -\frac{\frac{\partial \rho}{\partial c}(c_-, 0)}{\frac{\partial \rho}{\partial \lambda}(c_-, 0)} > 0.$$

This concludes the proof. □

The travelling wave solutions are stable with respect to small perturbations. More precisely, if we consider some initial condition which is sufficiently close to the travelling wave profile U, then the solution of (4) will tend to a translate of the travelling wave.

We indicate the proof of this statement which is basically a consequence of the work of Sattinger [12]. By the change of coordinates $\xi := x - ct$ we go to a comoving frame in which the travelling wave is a stationary solution. The linearization of (1) around this stationary solution is

$$\begin{aligned} v_t &= \varepsilon F_p\, v_{\xi\xi} + (F_u + F_{pu}U' + F_{pp}U'' + c)v_\xi + \varepsilon^{-1}\big(F_{uu}U' + F_{pu}U'' + g'(U)\big)v \\ &=: \mathcal{A}(\xi,\varepsilon)v. \end{aligned}$$

Here $F_u = F_u(U,U')$, $F_p = F_p(U,U')$, etc. are evaluated at the profile U and $\mathcal{A}(\xi,\varepsilon)$ is considered as an unbounded linear operator acting on $L^2(\mathbb{R},\mathbb{R})$.

To study the spectral properties of the linearized operator we perform another change of coordinates which makes the coefficient of the principal part of $\mathcal{A}(\xi,\varepsilon)$ constant. Using $\zeta = \zeta(\xi)$ with

$$\frac{d\zeta}{d\xi} = \frac{1}{\sqrt{F_p(U(\xi),U'(\xi))}}$$

we get for $w(t,\zeta) = v(t,\xi)$ the identity

$$w_\zeta \frac{d^2\zeta}{d\xi^2} + \frac{w_{\zeta\zeta}}{F_p(U(\xi),U'(\xi))} = v_{\xi\xi}.$$

In particular, w thus solves the equation

$$\begin{aligned} w_t &= \varepsilon w_{\zeta\zeta} + \left(F_u - \frac{\varepsilon}{2\sqrt{F_p}}(F_{pu}U' + F_{pp}U'') + c\right)w_\zeta \\ &\quad + \varepsilon^{-1}\big(F_{uu}U' + F_{pu}U'' + g'(U)\big)w \\ &=: \tilde{\mathcal{A}}(\zeta,\varepsilon)w. \end{aligned}$$

The operator $\tilde{\mathcal{A}}(\zeta,\varepsilon)$ fits in the framework discussed in [12] and [6]. In particular, to determine the essential spectrum it suffices to study the limiting (constant-coefficient) operators. Note that

$$\tilde{\mathcal{A}}_r v := \lim_{\zeta\to\infty} \tilde{\mathcal{A}}(\zeta,\varepsilon)w = \varepsilon w_{\zeta\zeta} + \big(F_u(u_r,0) + c\big)w_\zeta + \varepsilon^{-1}g'(u_r)w$$

and

$$\tilde{\mathcal{A}}_\ell w := \lim_{\zeta\to-\infty} \mathcal{A}(\zeta,\varepsilon)w = \varepsilon w_{\zeta\zeta} + \big(F_u(u_\ell,0) + c\big)w_\zeta + \varepsilon^{-1}g'(u_\ell)w.$$

The essential spectrum of $\tilde{\mathcal{A}}(\zeta,\varepsilon)$ is then contained in a domain bounded by two parabolas which intersect the real axis in $\frac{1}{\varepsilon}g'(u_\ell)$ and $\frac{1}{\varepsilon}g'(u_r)$ and which are open to the left. In particular, the essential spectrum of $\tilde{\mathcal{A}}(\zeta,\varepsilon)$ is strictly contained in the left half plane.

The remaining part of the spectrum is composed by isolated eigenvalues of finite multiplicity. It can be shown that 0 is the unique eigenvalue in the unstable half plane $\{\mu \in \mathbb{C}\colon \operatorname{Re}\mu \geqslant 0\}$; moreover, 0 is simple. Indeed, let us consider the eigenvalue problem

$$\tilde{\mathcal{A}}w + \mu w = \varepsilon w_{\zeta\zeta} + a(\zeta)\,w_\zeta + b(\zeta)\,w + \mu w = 0,$$

where

$$a(\zeta) = F_u(U,U') - \frac{\varepsilon}{2\sqrt{F_p}}\big(F_{pu}(U,U')\,U' + F_{pp}(U,U')\,U''\big) + c,$$
$$b(\zeta) = \varepsilon^{-1}\big(F_{uu}(U,U')\,U' + F_{pu}(U,U')\,U'' + g'(U)\big).$$

Then, the new variable $z = w\,\phi(\zeta)$, where $\phi(\zeta) := \exp(\frac{1}{2}\int a(s)\,ds)$ satisfies the selfadjoint equation (note that w decays sufficiently fast to guarantee that z is in $L^2(\mathbb{R},\mathbb{R})$)

$$\varepsilon z_{\zeta\zeta} - \frac{\varepsilon\phi_{\zeta\zeta}}{\phi}\,z + bz + \mu z = 0.$$

Hence all the eigenvalues are real. Moreover, multiplying by z and integrating

$$\int_{\mathbb{R}}\left[-\varepsilon z_\zeta^2 + \left(b - \frac{\varepsilon\phi_{\zeta\zeta}}{\phi}\right)z^2\right]d\zeta + \mu\int_{\mathbb{R}} z^2\,d\zeta = 0.$$

Since U' is solution of the original eigenvalue problem with $\mu = 0$, there exists a function ψ, never vanishing, such that

$$\varepsilon\psi_{\zeta\zeta} + \left(b - \frac{\varepsilon\phi_{\zeta\zeta}}{\phi}\right)\psi = 0,$$

hence, after an integration by parts,

$$-\mu\int_{\mathbb{R}} z^2\,d\zeta = \int_{\mathbb{R}}\left[-\varepsilon z_\zeta^2 - \frac{\varepsilon\psi_{\zeta\zeta}}{\psi}z^2\right]d\zeta = -\varepsilon\int_{\mathbb{R}}\psi^2\left[\frac{d}{d\zeta}\left(\frac{z}{\psi}\right)\right]^2 d\zeta \geqslant 0.$$

Immediately from this relation one can conclude that $\mu \geqslant 0$ and $\mu = 0$ implies $z = C\psi$ for some C, thus 0 is simple.

Since the linearized operator has 0 as a simple isolated eigenvalue while the rest of the spectrum is bounded away from the closed right half plane, this suffices to show orbital asymptotic stability of the travelling wave.

3. Layer formation

First of all we restrict out attention to initial datum u_0 satisfying stronger assumptions with respect to Theorem 1. More precisely assume $u_0 \in C^2(\mathbb{R},\mathbb{R})$ be such that

(H3′) $\{x \in \mathbb{R};\ u_0(x) = u_m\}$ is a finite set $\{x_1, x_2, \ldots, x_N\}$ with

$$m := \min_{1\leqslant i\leqslant N}\left|u_0'(x_i)\right| > 0 \quad \text{and} \quad \liminf_{x\to\pm\infty}\left|u_0(x) - u_m\right| > 0.$$

Note that (H3′) implies (H3).

Let us denote with $U = U(t;\sigma)$ the unique solution of

$$U_t = g(U), \qquad U(0;\sigma) = \sigma. \tag{9}$$

Then it is immediate to see that $U(\varepsilon^{-1}t;\sigma)$ solves $\varepsilon U_t = g(U)$.

Lemma 7. *Assume that $u_0 \in C^2(\mathbb{R},\mathbb{R})$ with $\|u_0\|_{C^2} \leqslant M_0$, satisfying* (H3′). *Then for any $k \in \mathbb{N}$ there exist τ_0, $\varepsilon_0(k)$ such that for $0 < \varepsilon < \varepsilon_0$,*

(i) $u^\varepsilon(x,t) \in [u_\ell - \varepsilon^k, u_r + \varepsilon^k]$ *for all $x \in \mathbb{R}$ and $t \geqslant k\tau_0\varepsilon \ln|\varepsilon|$.*

(ii)
$$\inf_{x\in\Omega_+^\varepsilon} u^\varepsilon(x,t_\varepsilon) \geqslant u_r - C\varepsilon|\ln\varepsilon|$$

and

$$\sup_{x\in\Omega_-^\varepsilon} u^\varepsilon(x,t_\varepsilon) \geqslant u_\ell + C\varepsilon|\ln\varepsilon|,$$

where

$$\begin{aligned}
\Omega_+^\varepsilon &:= \big\{x\in\mathbb{R}\colon\ |x-x_i| \geqslant \varepsilon|\ln\varepsilon|,\ u_0(x) > u_m\big\},\\
\Omega_-^\varepsilon &:= \big\{x\in\mathbb{R}\colon\ |x-x_i| \geqslant \varepsilon|\ln\varepsilon|,\ u_0(x) < u_m\big\},\\
t_\varepsilon &:= C^{-1}\varepsilon\ln|\ln\varepsilon|.
\end{aligned}$$

Proof. (i) Without restriction we may assume that $M_0 > u_r$. By (H2), it is possible to find a constant $\beta > 0$ such that

$$g(u) > -\beta(u-u_r)$$

holds for $u_r \leqslant u \leqslant M_0$. So, if

$$\overline{U}(t) = u_r + e^{-\beta t}(M_0 - u_r)$$

denotes the solution of the initial value problem

$$U' = -\beta(U-u_r), \qquad U(0) = M_0,$$

then $w(x,t) = \overline{U}(\varepsilon^{-1}t)$ is a supersolution. In particular, this implies that

$$\begin{aligned}
u^\varepsilon\big(x, k\tau_0\varepsilon|\ln\varepsilon|\big) \leqslant \overline{U}\big(k\tau_0|\ln\varepsilon|\big) &= u_r + e^{-\beta k\tau_0|\ln\varepsilon|}(M_0-u_r)\\
&= u_r + \varepsilon^{-\beta k\tau_0}(M_0-u_r).
\end{aligned}$$

Choosing τ_0 large and ε_0 small enough will make this expression smaller than ε^k for all $0 < \varepsilon \leqslant \varepsilon_0$.

The construction of a spatially homogeneous subsolution which proves that

$$u^\varepsilon\big(x, k\tau_0\varepsilon|\ln\varepsilon|\big) \geqslant u_\ell - \varepsilon^k$$

is completely analogous.

(ii) Let $U^\lambda = U^\lambda(t;\sigma)$ be the solution of (9) with g replaced by $g^\lambda = g + \lambda$ for some $\lambda \in \mathbb{R}$ and set

$$w^\lambda(x,t) = U^\lambda\big(\varepsilon^{-1}t; u_0(x) - \varepsilon M t\big).$$

Since

$$\varepsilon w_t = g(U^\lambda) + \lambda - \varepsilon M U^\lambda_\sigma, \quad w_x = u_0'\,U^\lambda_\sigma, \quad w_{xx} = (u_0')^2\,U^\lambda_{\sigma\sigma} + u_0''\,U^\lambda_\sigma,$$

there holds

$$\mathcal{L}^\varepsilon w^\lambda = \frac{\lambda}{\varepsilon} - \varepsilon F_p (u_0')^2\,U^\lambda_{\sigma\sigma} - \big[\varepsilon F_p\,u_0'' + F_u u_0' + M\big]\,U^\lambda_\sigma, \tag{10}$$

where F_p and F_u are calculated at $(w^\lambda, \varepsilon u_0' U^\lambda_\sigma)$.

In order to prove that $\mathcal{L}^\varepsilon w^\lambda$ is positive (negative) for λ positive (negative) and ε small enough, it is necessary to estimate U^λ_σ and $U^\lambda_{\sigma\sigma}$. The functions U^λ_σ and $U^\lambda_{\sigma\sigma}$ satisfy, respectively,

$$\begin{cases} v_t = g'(U^\lambda)v, \\ v(0) = 1, \end{cases} \qquad \text{and} \qquad \begin{cases} z_t = g'(U^\lambda)z + g''(U^\lambda)v^2, \\ z(0) = 0, \end{cases}$$

so that

$$U^\lambda_\sigma\left(\frac{t}{\varepsilon};\sigma\right) = \exp\left(\frac{1}{\varepsilon}\int_0^t g'\left(U^\lambda\left(\frac{s}{\varepsilon}\right)\right)ds\right)$$

and

$$U^\lambda_{\sigma\sigma}\left(\frac{t}{\varepsilon};\sigma\right) = \frac{1}{\varepsilon}\int_0^t \exp\left(\frac{1}{\varepsilon}\int_0^{t-s} g'\left(U^\lambda\left(\frac{r}{\varepsilon}\right)\right)dr\right)g''\left(U^\lambda\left(\frac{s}{\varepsilon}\right)\right)\left[U^\lambda_\sigma\left(\frac{s}{\varepsilon}\right)\right]^2 ds.$$

Hence

$$0 \leqslant U^\lambda_\sigma(t/\varepsilon;\sigma) \leqslant e^{Ct/\varepsilon}, \qquad \left|U^\lambda_{\sigma\sigma}(t/\varepsilon;\sigma)\right| \leqslant Ce^{Ct/\varepsilon}, \tag{11}$$

for some constant $C > 0$ depending on $\sup|g'|$, $\sup|g''|$, but independent of ε and σ.

(Note also that, for $g(\sigma) + \lambda \neq 0$, by the change of variable $u = U^\lambda(s,\sigma)$, we get $U^\lambda_\sigma = g(U^\lambda)\,g^{-1}(\sigma)$ and $U^\lambda_{\sigma\sigma} = [g'(U^\lambda) - g'(\sigma)]\cdot g(U^\lambda)g^{-2}(\sigma)$; if $g(\sigma) + \lambda = 0$, then $U^\lambda_\sigma = \exp(g'(\sigma)t)$ and $U^\lambda_{\sigma\sigma} = g''(\sigma)\,t\,\exp(g'(\sigma)t)$.)

Let us choose $\lambda = \lambda_\pm := \pm C\varepsilon|\ln\varepsilon|$. With this choice, from (11) we get for $0 \leqslant t \leqslant t_\varepsilon = C^{-1}\varepsilon\ln|\ln\varepsilon|$,

$$\mathcal{L}^\varepsilon w^{\lambda_-} \leqslant \frac{\lambda_-}{\varepsilon} + Ce^{Ct/\varepsilon} + (C-M)U^\lambda_\sigma \leqslant -C|\ln\varepsilon| + C|\ln\varepsilon| + (C-M)U^\lambda_\sigma.$$

Hence, for M sufficiently large $\mathcal{L}^\varepsilon w^{\lambda_-} \leqslant 0$ and w^{λ_-} is a subsolution. Similarly, for ε small w^{λ_+} is a supersolution of the problem for all $t \in [0, t_\varepsilon]$.

Since there holds $w^{\lambda_-}(x,t_\varepsilon) \leqslant u^\varepsilon(x,t_\varepsilon) \leqslant w^{\lambda_+}(x,t_\varepsilon)$ we have

$$\begin{aligned} &U^{\lambda_-}\big(t_\varepsilon/\varepsilon; u_0(x) - M\varepsilon^2\ln|\ln\varepsilon|\big) \leqslant u^\varepsilon(x,t_\varepsilon) \\ &\quad \leqslant U^{\lambda_+}\big(\ln|\ln\varepsilon|; u_0(x) - M\varepsilon^2\ln|\ln\varepsilon|\big), \end{aligned}$$

where $\lambda_\pm = \pm C\varepsilon|\ln\varepsilon|$.

We now indicate that if $u_0(x) > \varepsilon|\ln\varepsilon|$ then

$$\left|U^{\lambda_-}\big(\ln|\ln\varepsilon|; u_0(x) - M\varepsilon^2\ln|\ln\varepsilon|\big) - u^\lambda_r\right| \leqslant \varepsilon \tag{12}$$

for ε sufficiently small. This in turn implies that

$$\begin{aligned} &\left|U^{\lambda_-}\big(\ln|\ln\varepsilon|; u_0(x) - M\varepsilon^2\ln|\ln\varepsilon|\big) - u_r\right| \\ &\quad \leqslant \left|U^{\lambda_-}\big(\ln|\ln\varepsilon|; u_0(x) - M\varepsilon^2\ln|\ln\varepsilon|\big) - u^\lambda_r\right| + |u^\lambda_r - u_r| \\ &\quad \leqslant \varepsilon + C\varepsilon|\ln\varepsilon| \leqslant \tilde{C}\varepsilon|\ln\varepsilon|. \end{aligned}$$

Inequality (12) is a consequence of some linear estimates: By, (H2), there is a constant $\alpha > 0$ such that

$$g(u) \geqslant \alpha(u - u_m) \quad \text{for } u_m \leqslant u \leqslant \frac{1}{2}(u_m + u_r),$$
$$g(u) \geqslant \alpha(u - u_m) \quad \text{for } \frac{1}{2}(u_m + u_r) \leqslant u \leqslant u_r.$$

We know from (H3′) that there is some constant $\gamma > 0$ such that $u_0(x) > u_m + \gamma\varepsilon|\ln\varepsilon|$ if $|x - x_i| > \varepsilon|\ln\varepsilon|$ and $u_0(x) > u_m$. For ε sufficiently small we then have $u_0(x) - \varepsilon M t_\varepsilon > u_m + \frac{\gamma}{2}\varepsilon|\ln\varepsilon|$. A straightforward comparison argument now proves our claim.

Dealing similarly with the other cases in the same way, we get the conclusion. □

Since for any $k \in \mathbb{N}$ there is some $\varepsilon_0 = \varepsilon_0(k)$ such that $t_\varepsilon > k\tau_0\varepsilon|\ln\varepsilon|$ holds for $0 < \varepsilon < \varepsilon_0$ we can combine part (i) and (ii) of the previous lemma:

Corollary 8. *At time $t = t_\varepsilon$ we have $u_r - C\varepsilon|\ln\varepsilon| \leqslant u^\varepsilon(x, t_\varepsilon) \leqslant u_r + \varepsilon^k$ if $u_0(x) < u_m$ and $|x - x_i| > \varepsilon|\ln\varepsilon|$ for all i. Analogously, we have $u_\ell - \varepsilon^k \leqslant u^\varepsilon(x, t_\varepsilon) \leqslant u_\ell + C\varepsilon|\ln\varepsilon|$ if $u_0(x) > u_m$ and $|x - x_i| > \varepsilon|\ln\varepsilon|$.*

Remark 9. Chen [1] uses a slightly more complicated modification of g to show that in the case $F = p$ after a time of order $\mathcal{O}(\varepsilon|\ln\varepsilon|)$ the solution outside some $\mathcal{O}(\sqrt{\varepsilon}\,|\ln\varepsilon|)$-neighborhood of the zeroes of u_0 is even ε^k-close to u_ℓ or u_r. With minor modifications his proof applies to Eq. (2) as well. In our approach the layer is localized more accurately in space although our estimates on the closeness of the solution to the equilibrium state u_ℓ and u_r are less precise than Chen's.

4. The singular limit $\varepsilon \to 0$

In order to construct sub- and supersolution it is useful to shift the reaction function by a constant amount. To this aim, let us introduce

$$g^\lambda(s) := g(s) + \lambda.$$

For λ is sufficiently small, g^λ has the same structure of zeros of g, hence we set

$$g^\lambda(s) = 0 \quad \Leftrightarrow \quad s \in \{u_-^\lambda < u_0^\lambda < u_+^\lambda\}.$$

Next, let $(U_+^\lambda(\cdot), c_+^\lambda)$ denote a pair profile/speed satisfying

$$F_p(U, U')U'' + \big(c + F_u(U, U')\big)U' + g(U) = 0, \quad U(\pm\infty) = u_\pm^\lambda,$$

where (by definition) $u_-^\lambda < u_+^\lambda$ are the stable zeros of $g^\lambda(\cdot)$. Similarly $(U_-^\lambda(\cdot), c_-^\lambda)$ denotes a solution of the same equation with reversed asymptotic states $U_-(-\infty) = u_+^\lambda$ and $U_-(+\infty) = u_-^\lambda$. To fix the profiles of the waves, we choose $U_\pm^\lambda$ so that

$$U_\pm^\lambda(0) = 0.$$

It follows from the assumption (H2) on g that there exist constants $C, \nu > 0$ (independent of λ small) such that

$$\begin{aligned} \left|U^\lambda_\pm(\xi) - u^\lambda_\mp\right| &\leqslant Ce^{-\nu\xi} \quad \forall \xi > 0, \\ \left|U^\lambda_\pm(\xi) - u^\lambda_\pm\right| &\leqslant Ce^{+\nu\xi} \quad \forall \xi < 0, \end{aligned} \tag{13}$$

and

$$\left|\left(U^\lambda_\pm\right)'(\xi)\right| \leqslant Ce^{-\nu|\xi|} \quad \forall \xi, \tag{14}$$

4.1. *Construction of super- and subsolutions with two layers*

Let $\theta = \theta(x,t,\varepsilon)$ be a C^2-function with values in $[0,1]$ and set

$$W(x,t) = (1-\theta)U^\lambda_-\left(\frac{x - c^\lambda_- t - a_-}{\varepsilon}\right) + \theta\, U^\lambda_+\left(\frac{x - c^\lambda_+ t - a_+}{\varepsilon}\right) \tag{15}$$

for some $a_- < a_+$. Then (omitting for shortness the index λ)

$$\begin{aligned} \mathcal{L}^\varepsilon W = {} & (\theta_t - \varepsilon F_p \theta_{xx} - F_u\theta_x)(U_+ - U_-) - 2F_p\theta_x(U'_+ - U'_-) \\ & + \frac{\theta}{\varepsilon}\big[(F_p(U_+, U'_+) - F_p)U''_+ + \big(F_u(U_+, U'_+) - F_u\big)U'_+ \\ & + \big(g(U_+) - g(W)\big)\big] + \frac{1-\theta}{\varepsilon}\big[\big(F_p(U_-, U'_-) - F_p\big)U''_- \\ & + \big(F_u(U_-, U'_-) - F_u\big)U'_- + \big(g(U_-) - g(W)\big)\big] + \frac{\lambda}{\varepsilon}, \end{aligned}$$

where $F_p = F_p(W, \varepsilon W_x)$, etc. Note that

$$\begin{aligned} U_+ - W &= (1-\theta)(U_+ - U_-), \\ U_- - W &= -\theta(U_+ - U_-), \\ U'_+ - \varepsilon W_x &= (1-\theta)(U'_+ - U'_-) - \varepsilon\theta_x(U_+ - U_-), \\ U'_- - \varepsilon W_x &= -\theta(U'_+ - U'_-) - \varepsilon\theta_x(U_+ - U_-). \end{aligned}$$

Hence

$$\mathcal{L}^\varepsilon W = \frac{\lambda}{\varepsilon} \quad \forall (x,t) \in \operatorname{Int}\{\theta = 0\} \cup \operatorname{Int}\{\theta = 1\}. \tag{16}$$

Thus, in the regions where θ is constant it suffices to have $\lambda < 0$ for a subsolution ($\lambda > 0$ for a supersolution).

We will now be more specific in our choices: let $\theta(t,x,\varepsilon) := \chi(\frac{x-\eta(t)}{\varepsilon|\ln\varepsilon|})$, where $\chi \in C^2(\mathbb{R}, [0,1])$ satisfies

(i) $\chi(\xi) \equiv 0$ for $\xi \leqslant -1$,
(ii) $\chi(\xi) \equiv 1$ for $\xi \geqslant 1$ and
(iii) $\|\chi\|_{C^2} \leqslant 2$,

and $|\eta'|_{C^1}$ is bounded. Then, from smoothness of f and g, boundedness of $U_\pm, U'_\pm$, and the properties of χ we can deduce that

$$\begin{aligned}\mathcal{L}^\varepsilon W &\leqslant \frac{C}{\varepsilon}\left[\left(\frac{\|\chi'\|}{|\ln\varepsilon|}(1+\|\eta'\|)+\frac{\|\chi''\|}{|\ln\varepsilon|^2}\right)|U_+-U_-|+|U_+-U_-|+|U'_+|+|U'_-|\right]\\ &\quad+\frac{\lambda}{\varepsilon}\\ &\leqslant \frac{C}{\varepsilon}\big(|U_+-U_-|+|U'_+|+|U'_-|\big)+\frac{\lambda}{\varepsilon} \qquad (17)\end{aligned}$$

for some $C>0$, depending on χ, f, g but not on ε and λ.

Lemma 10. *Let $a_+ > a_-$ and set $\lambda = C\varepsilon|\ln\varepsilon|$,*

$$T_\varepsilon := \begin{cases} +\infty & \text{if } c^\lambda_- \leqslant c^\lambda_+,\\ \frac{a_+-a_-}{c^\lambda_- - c^\lambda_+} - K\varepsilon|\ln\varepsilon| & \text{if } c^\lambda_- > c^\lambda_+,\end{cases}$$

where $K = 2(\frac{2}{\nu}+1)(c^\lambda_- - c^\lambda_+)^{-1}$ and ν as in (13)–(14). *Moreover, as above let $\theta(t,x,\varepsilon) = \chi(\frac{x-\eta(t)}{\varepsilon|\ln\varepsilon|})$ with*

$$\eta(t) = \frac{a_+ + a_-}{2} + \frac{c_+ + c_-}{2}t.$$

Then for ε sufficiently small

$$W^\varepsilon(x,t) = (1-\theta)U^\lambda_-\left(\frac{x - c^\lambda_- t - a_-}{\varepsilon}\right) + \theta\, U^\lambda_+\left(\frac{x - c^\lambda_+ t - a_+}{\varepsilon}\right)$$

is a supersolution for $0 \leqslant t \leqslant T_\varepsilon$. For $a_- \geqslant a_+$, let $\lambda = -C\varepsilon|\ln\varepsilon|$ and define T_ε as above with $\geqslant$ in place of $\leqslant$ and (*in place of*)*. Then, with the same choice of θ, for ε sufficiently small*

$$W^\varepsilon(x,t) = \theta\, U^\lambda_-\left(\frac{x - c^\lambda_- t - a_-}{\varepsilon}\right) + (1-\theta)\, U^\lambda_+\left(\frac{x - c^\lambda_+ t - a_+}{\varepsilon}\right)$$

is a subsolution for $0 \leqslant t \leqslant T_\varepsilon$.

Remark 11. With Lemma 10, we end up with sub- and supersolutions representing (approximately) a pattern of the two waves U^λ_+ and U^λ_-. If the time of existence T_ε is finite, the pattern is interacting (giving raise to cancellation of waves), if T_ε is infinite, the pattern is noninteracting, since the waves are diverging.

Proof. We only deal with the supersolution case.

We need to estimate $U_+ - U_-$ and $U'_\pm$ in appropriate regions of the half space $\{t > 0\}$. Set

$$D_\varepsilon := \left\{(x,t)\colon\ a_- + c^\lambda_- t + \frac{2}{\nu}\varepsilon|\ln\varepsilon| < x < a_+ + c^\lambda_+ t - \frac{2}{\nu}\varepsilon|\ln\varepsilon|\right\}. \qquad (18)$$

Then, from (13)–(14), it follows that

$$|U_+ - U_-| + |U_+'| + |U_-'| \leqslant C \exp\left\{-\frac{\nu}{\varepsilon}\frac{2}{\nu}\varepsilon|\ln\varepsilon|\right\} = C\varepsilon^2 \quad \text{in } D_\varepsilon \tag{19}$$

for some $C > 0$ independent of ν, ε, λ.

Therefore, collecting (16), (17) and (19), we get

$$\mathcal{L}^\varepsilon W^\varepsilon \geqslant \frac{C}{\varepsilon} C\varepsilon^2 + \frac{\lambda}{\varepsilon}, \quad (x,t) \in D_\varepsilon.$$

Obviously, for $\lambda = C\varepsilon|\ln\varepsilon|$ and ε small enough, $\mathcal{L}^\varepsilon$ is positive in D_ε. Outside of D_ε, inequality $\mathcal{L}^\varepsilon W^\varepsilon \geqslant 0$ holds due to (16). Hence, the function W^ε is a subsolution in $(0, T_\varepsilon) \times \mathbb{R}$. □

In the limit $\varepsilon \to 0^+$, W^ε tends to a piecewise constant function with constants states separated by straight lines emanating from the points $a_\pm$ and with slopes $c_\pm$. More precisely, if $a_- < a_+$, then

$$\lim_{\varepsilon\to 0} W^\varepsilon(x,t) = \begin{cases} u_\ell & \text{if } a_- + c_- t < x < a_+ + c_+ t, \\ u_r & \text{otherwise.} \end{cases}$$

Similarly for the case $a_+ < a_-$.

4.2. *Proof of Theorem 1*

First of all, let us assume $u_0 \in C^2(\mathbb{R}, \mathbb{R})$ with $\|u_0\|_{C^2} \leqslant M_0$, satisfying (H3′).

Since there are only two different wave speeds, there will be no interaction of more than two fronts at the same point. Without loss of generality, we can assume that $c_- \leqslant c_+$.

Thanks to Lemma 7, we know that at time $t_\varepsilon = C^{-1}\varepsilon \ln|\ln\varepsilon|$ sharp layers have been formed at the points $x_1, x_2, \ldots, x_N$. Set $x_0 = -\infty$ and $x_{N+1} = +\infty$. These $N+2$ values divide the real line in $N+1$ open intervals; for each of these intervals we will now construct a sub- and a supersolution W_i.

(i) If $u(x, t_\varepsilon) < u_m$ at $\frac{x_{i-1}+x_i}{2}$, we choose a supersolution $\overline{W}_i^\varepsilon$ as in Lemma 10 with $a_- = x_{i-1}$ and $a_+ = x_i$. Note that in case $x_{i-1} = -\infty$ or $x_i = +\infty$, the function $\overline{W}_i^\varepsilon$ possesses only one layer. The subsolution $\underline{W}_i^\varepsilon$ is defined to be equal to $u_\ell - \varepsilon^k$.

(ii) If $u(x, t_\varepsilon) > u_m$ at $\frac{x_{i-1}+x_i}{2}$, then we choose a subsolution $\underline{W}_i^\varepsilon$ with $a_+ = x_{i-1}$ and $a_- = x_i$. The supersolution $\overline{W}_i^\varepsilon$ is defined to be equal to $u_r + \varepsilon^k$.

Note that the supersolution are always defined for any time $t > t_\varepsilon$, while some subsolutions (with x_{i-1} and x_i both finite) are defined locally in time.

By Lemma 7 and Corollary 8, the supersolutions $\overline{W}_i^\varepsilon$ satisfies $u(x, t_\varepsilon) \leqslant \overline{W}_i^\varepsilon(x, t_\varepsilon)$ for any x and the subsolutions $\underline{W}_i^\varepsilon$ satisfies $u(x, t_\varepsilon) \geqslant \underline{W}_i^\varepsilon(x, t_\varepsilon)$. Applying comparison principle (see [11, Chapter 3, Section 7]), we deduce

$$\underline{W}^\varepsilon(x,t) \leqslant u^\varepsilon(x,t) \leqslant \overline{W}^\varepsilon(x,t) \quad \forall (x,t),\ t > t_\varepsilon,$$

where

$$\underline{W}^\varepsilon(x,t) := \max\left\{\underline{W}_i^\varepsilon(x,t)\colon\ \underline{W}_i^\varepsilon \text{ is defined at time } t\right\},$$
$$\overline{W}^\varepsilon(x,t) := \min\left\{\overline{W}_i^\varepsilon(x,t)\right\}.$$

Fix (x,t) not belonging to any of the straight lines $x_i + c_\pm t$, then it is contained in (at least) one region either of the form

$$\{(x,t):\ x_{i-1} + c_- t < x < x_i + c_+ t\},$$

or

$$\{(x,t):\ x_{i-1} + c_+ t < x < x_i + c_- t\}.$$

In the first case, the supersolution $\overline{W}_i^\varepsilon$ converges to u_ℓ on the set $\{x_{i-1} + c_- t < x < x_i + c_+ t\}$. Since the subsolution $\underline{W}_i^\varepsilon = u_\ell - \varepsilon^k$ tends to u_ℓ, we have pointwise convergence. In the second case, the subsolution $\underline{W}_i^\varepsilon$ is defined up to a finite time, always greater than t, and it converges to u_r. This, together with the supersolution $\overline{W}_i^\varepsilon = u_r + \varepsilon^k$, gives the result for $u_0 \in C^2(\mathbb{R},\mathbb{R})$ with bounded second derivative and satisfying (H3′).

The general case can be dealt with by comparison principle, using smoothed version of the initial datum u_0. Let u_0 be as in Theorem 1. In order to simplify the presentation, let us assume that there exists $\{x_1 < \cdots < x_{2n+1}\} \subset \mathbb{R}$ for which, for any $\delta \in (0, \frac{1}{2}\min(|x_2 - x_1|, \ldots, |x_n - x_{n-1}|)$, and for any $k = 0, \ldots, n$, there holds

$$\operatorname{ess\,sup}\{u_0(x):\ x \in I_{2k+1}^\delta\} < u_m < \operatorname{ess\,inf}\{u_0(x):\ x \in I_{2k}^\delta\},$$

where $I_k^\delta = (x_k + \delta, x_{k+1} - \delta)$, $x_0 = -\infty$ and $x_{2n+2} = +\infty$. Then the conclusion of Theorem 1 holds.

Indeed, fix $\delta > 0$ and let $\mu = \mu(\delta) > 0$ such that for any $k = 0, \ldots, n$,

$$\operatorname{ess\,sup}\{u_0(x):\ x \in I_{2k+1}^\delta\} \leqslant u_m - \mu < u_m < u_m + \mu \leqslant \operatorname{ess\,inf}\{u_0(x):\ x \in I_{2k}^\delta\}.$$

Then choose $\bar{u}_0^\delta, \underline{u}_0^\delta \in C^2(\mathbb{R},\mathbb{R})$ satisfying:

(i) for $k = 0, \ldots, n$,

$$\underline{u}_0^\delta(x) = \begin{cases} -\|u_0\|_{L^\infty}, & x \in (x_{2k+1} - \delta, x_{2k+2} + \delta), \\ u_m + \mu, & x \in (x_{2k} + 2\delta, x_{2k+1} - 2\delta), \end{cases}$$

$$\bar{u}\delta_0(x) = \begin{cases} \|u_0\|_{L^\infty}, & x \in (x_{2k} - \delta, x_{2k+1} + \delta), \\ u_m - \mu, & x \in (x_{2k+1} + 2\delta, x_{2k} - 2\delta); \end{cases}$$

(ii) there hold

$$\frac{d}{dx}\underline{u}_0^\delta < 0 \quad \text{in } \bigcup_{k=0}^n (x_{2k+1} - 2\delta, x_{2k+1} - \delta),$$

$$\frac{d}{dx}\underline{u}_0^\delta(x) > 0 \quad \text{in } \bigcup_{k=0}^n (x_{2k} + \delta, x_{2k} + 2\delta),$$

$$\frac{d}{dx}\bar{u}_0^\delta(x) < 0 \quad \text{in } \bigcup_{k=0}^n (x_{2k+1} + \delta, x_{2k+1} + 2\delta),$$

$$\frac{d}{dx}\bar{u}_0^\delta(x) > 0 \quad \text{in } \bigcup_{k=0}^n (x_{2k} - 2\delta, x_{2k} - \delta);$$

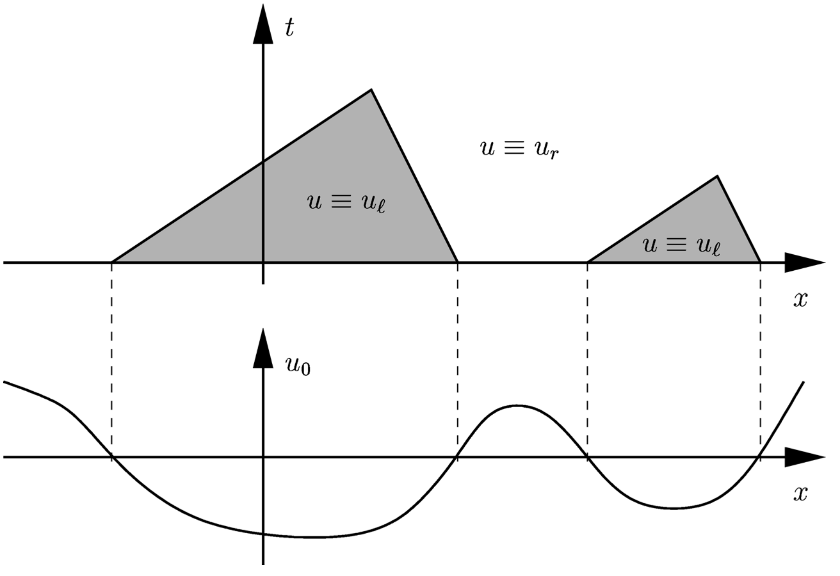

Fig. 3. The limit $u(t,x) = \lim_{\varepsilon \to 0} u^\varepsilon(t,x)$.

(iii) there exists positive M_0 (possibly depending on δ) such that

$$\|\bar{u}_0^\delta\|_{C^2}, \|\underline{u}_0^\delta\|_{C^2} \leqslant M_0.$$

The functions $\bar{u}_0^\delta, \underline{u}_0^\delta$ enjoy assumptions (H3′) and, additionally, satisfy

$$\underline{u}_0^\delta(x) \leqslant u_0(x) \leqslant \bar{u}_0^\delta(x) \quad \text{a.e. in } \mathbb{R}.$$

Then, by comparison principle, for almost any $t > 0$,

$$\underline{u}^{\delta,\varepsilon}(x,t) \leqslant u^\varepsilon(x,t) \leqslant \bar{u}^{\delta,\varepsilon}(x,t) \quad \text{for almost any } (x,t).$$

By construction there holds

$$\lim_{\delta \to 0^+} \lim_{\varepsilon \to 0^+} \underline{u}^{\delta,\varepsilon}(x,t) = \lim_{\delta \to 0^+} \lim_{\varepsilon \to 0^+} \bar{u}^{\delta,\varepsilon}(x,t) \quad \text{for almost any } (x,t),$$

and the common limit is given a.e. by the limit described in the statement of Theorem 1. Hence, also $\lim_{\varepsilon \to 0} u^\varepsilon(x,t)$ exists for a.e. (x,t) and coincide a.e. with the same function.

Figure 3 shows the structure of the limiting solution. □

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